THE HODGE DECOMPOSITION

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1. The Musical Isomorphisms and induced metrics

Given a smooth Riemannian manifold (X, g), TX will denote the tangent bundle; T^*X the cotangent bundle. The additional structure provided by the metric g allows us to have some sense of "duality" between vector fields and differential forms.

Definition 1.1 (The Musical Isomorphisms)**.** Let $\omega \in T^*X$. We have an association $\omega \mapsto \omega^\sharp$ with

$$g(\omega^\sharp, Z) = \omega(Z)$$

for $Z \in TX$. Similarly, given $Z \in TX$, we have the association $Z \mapsto Z^\flat$ with

$$Z^\flat(Y) = g(Z, Y)$$

for $Y \in TX$.

Proposition 1.2. *The above maps constitute an isomorphism $T^*X \cong TX$.*

Proof. The above maps are inverses for each other, so this is a tautology. $\square$

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Suppose now that X is compact with metric g. Let $\Omega^k_{X,\mathbb{R}}$ denote the vector bundle of k-forms over X. The musical isomorphisms give an induced metric structure to $\Omega^k_{X,x}$, denoted $(\cdot,\cdot)_x$, by setting

$$(\omega,\sigma)_x := g_x(\omega^\sharp,\sigma^\sharp)$$

By convention it is assumed that ω, σ are evaluated at $x \in X$. We can now go further to define the L^2 metric $(\cdot,\cdot)_{L^2}$ on the space $A^k(X)$ of differential k-forms on X:

$$(\omega,\sigma)_{L^2} := \int_X (\omega,\sigma)\mathrm{vol}$$

with (ω,σ) being the function on X defined by sending $x \mapsto (\omega,\sigma)_x$.

2. The Hodge $*$ Operator

Suppose again that X is a compact smooth manifold with metric g, and additionally assume $\dim X = n$. We have an isomorphism

$$p : \bigwedge^{n-k} \Omega_{X,x} \to \mathrm{Hom}\left(\bigwedge^k \Omega_{X,x}, \bigwedge^n \Omega_{X,x}\right)$$

defined by $p(\omega)(\sigma) = \sigma \wedge \omega$. Similarly, we get another isomorphism

$$m : \bigwedge^k \Omega_{X,x} \to \mathrm{Hom}\left(\bigwedge^k \Omega_{X,x}, \mathbb{R}\right)$$

with $m(\Omega)(\sigma) = (\omega,\sigma)_x$. Then, noting that $\bigwedge^n \Omega_{X,x} \cong \mathbb{R}$ via the isomorphism sending $r \in \mathbb{R} \mapsto r \cdot \mathrm{vol}$, consider the isomorphism

$$p^{-1} \circ m : \bigwedge^k \Omega_{X,x} \to \bigwedge^{n-k} \Omega_{X,x}$$

Denote this map by $*_x$; this should vary smoothly with x whenever g is a smooth metric. By construction, we see that

$$p(*_x\omega)(\sigma) = \sigma \wedge *_x\omega$$

and, as $p^{-1}*_x = m$, we also see

$$p^{-1}(*_x\omega)(\sigma) = m(\omega)(\sigma) = (\omega, \sigma)\mathrm{vol}$$

Definition 2.1. Let $*$ denote the isomorphism of vector bundles

$$* : \Omega^k_{X,\mathbb{R}} \to \Omega^{n-k}_{X,\mathbb{R}}$$

constructed as above; this also induces a morphism on the smooth sections of the above vector bundles

$$* : A^k(X) \to A^{n-k}(X)$$

This map is called the Hodge $*$ operator.

Proposition 2.2. *For* $\omega,\ \sigma \in A^k(X)$*,*

$$(\omega, \sigma)_{L^2} = \int_X \alpha \wedge *\beta$$

Proof. By definition. □

3. Kähler Manifolds and Complexification of $*$

Definition 3.1. A Kähler Manifold is a symplectic manifold (X, ω) equipped with a compatible almost complex structure J (that is, $J^2 = -1$). More precisely,

$$g(X, Y) := \omega(X, JY)$$

is a Riemannian metric.

The symplectic form ω may also be referred to as a Hermitian metric. In an identical manner to before, we have an induced Hermitian metric on the vector bundles.

Now, given real valued vector bundles, we may complexify by formally tensoring with $\mathbb{C}$ (over $\mathbb{R}$). The previously induced metrics are easily extended to Hermitian metrics on our complexified bundles $\Omega^k_{X,\mathbb{C}}$. The complexified Hodge $*$ now satisfies

$$(\omega, \sigma)_x \text{vol} = \omega \wedge \overline{*\sigma}$$

on each fiber, and by smoothness of our metric may be extended in a smooth manner to $\Omega^k_{X,\mathbb{C}}$.

4. Formal Adjoints and the Laplacian

We have a map

$$d : A^{k-1}(X) \to A^k$$

Where d denotes the standard exterior derivative. The introduction of the L^2 metric allows one to ask questions on the existence of adjoints; that is, a map $d^* : A^k(X) \to A^{k-1}(X)$ satisfying

$$(d\alpha, \beta)_{L^2} = (\alpha, d^*\beta)_{L^2}$$

Recall that for a complex manifold X, we have operators ∂ and $\overline{\partial}$ with $d = \partial + \overline{\partial}$. The adjoints for these operators have a particularly nice form:

Proposition 4.1. *With respect to the Hermitian L^2 metric, we have formal adjoints*

$$\partial^* = - * \overline{\partial} *$$

$$\overline{\partial}^* = - * \partial *$$

Proof. Recalling that ∂ and $\overline{\partial}$ are graded derivations, we have:

$$\begin{aligned}
(\overline{\partial}\alpha, \beta)_{L^2} &= \int_X \overline{\partial}\alpha \wedge \overline{*\beta} \\
&= - \int_X \overline{\partial}\big(\alpha \wedge \overline{*\beta}\big) - \int_X (-1)^{|\alpha|} \alpha \wedge \partial\overline{*\beta} \\
&= - \int_X (-1)^{|\alpha|} \alpha \wedge \overline{*}\big(\overline{*^{-1}\partial * \beta}\big)
\end{aligned}$$

Recall that $*^{-1} = (-1)^{k(2n-k)}*$, where in this case $2n - |\alpha| = k$. Whence we see

$$\begin{aligned} -\int_X (-1)^{|\alpha|}\alpha \wedge \overline{*}\left(\overline{*^{-1}\partial * \beta}\right) &= \int_X (-1)^{|\alpha|}\alpha \wedge \overline{*}\left((-1)^{|\alpha|}\overline{*\partial * \beta}\right) \\ &= -\int_X \alpha \wedge \overline{*}\left(\overline{*\partial * \beta}\right) \\ &= (\alpha, - * \partial * \beta)_{L^2} \end{aligned}$$

The computation for ∂ is essentially identical. ☐

Definition 4.2. For a Riemannian manifold (M, g), define the operator

$$\Delta_d := d^*d + dd^*$$

Δ_d is called the Laplacian.

One easily sees that a form is annihilated by the Laplacian if and only if it is both closed and coclosed. More precisely,

Proposition 4.3. *If X is compact,*

$$(\alpha, \Delta_d \alpha)_{L^2} = (d\alpha, d\alpha)_{L^2} + (d^*\alpha, d^*\alpha)_{L^2}$$

In particular, $\operatorname{Ker} \Delta_d = \operatorname{Ker} d \cap \operatorname{Ker} d^*$.

Proof. By definition. ☐

Definition 4.4. Any form annihilated by the Laplacian Δ_d is called a *harmonic form* (or a Δ_d-harmonic form, for clarity).

5. Elliptic Partial Differential Operators

Let E and F denote real or complex smooth vector bundles over a manifold M. Let $\underline{C^\infty}$ denote the constant sheaf assigning to a vector bundle the smooth sections, and suppose that

$$P : \underline{C^\infty}(E) \to \underline{C^\infty}(F)$$

is a $\mathbb{R}$ or $\mathbb{C}$-linear morphism of sheaves.

Definition 5.1. P is a differential operator of order k if for open sets U with coordinates $\{x_i\}_{i=1}^n$ and trivializations

$$E|_U \cong U \times \mathbb{R}^p \quad F|_U \cong U \times \mathbb{R}^q$$

we have that

$$P\big((\alpha_1, \dots, \alpha_n)\big) = (\beta_1, \dots, \beta_n)$$

with

$$\beta_i = \sum_{I,j} P_{I,i,j} \frac{\partial \alpha_j}{\partial x_I}$$

with $P_{I,i,j} \in C^\infty(M)$ and vanishing for $|I| > k$. More succinctly, in multi-index notation one may write for $u \in C^\infty(M, E)$,

$$Pu(x) = \sum_{|\alpha| \leqslant k} P_\alpha(x) D^\alpha u(x)$$

Essentially a differential operator is something that locally induces a partial differential equation in the local coordinates of a given chart. We can associate the differential operator P to a mapping σ_P such that for $x \in M$,

$$\begin{aligned} \sigma_P(x, -) :& T^*_{X,x} \to \mathrm{Hom}(E_x, F_x) \\ & \xi \mapsto \sum_{|\alpha| = k} P_\alpha(x) \xi^\alpha \end{aligned}$$

Definition 5.2. The map σ_P such that

$$\sigma_P(x, -) : T^*_{X,x} \to \mathrm{Hom}(E_x, F_x)$$

is called the *symbol* of the differential operator P.

Definition 5.3. A differential operator P is called *elliptic* if the map $\xi \mapsto \sigma_P(x, \xi)$ is injective for all $x \in X$.

Lemma 5.4. *Let* $P : \underline{C^\infty}(E) \to \underline{C^\infty}(F)$*. Then, the formal adjoint* $P^* : \underline{C^\infty}(F) \to \underline{C^\infty}(E)$ *exists, is unique, and satisfies*

$$(\alpha, P\beta)_{L^2} = (P^*\alpha, \beta)_{L^2} \text{ for all } \alpha \in C^\infty(F),\ \beta \in C^\infty(E)$$

Proof. It is enough to define P^* locally, since we may get the global result by merely extending by a partition of unity. Likewise, uniqueness follows from noting that smooth functions with compact support are dense in the L^2 metric. Since P^* is defined locally, it is unique on compactly supported functions.

Choose local coordinates $x_1, \dots, x_n$ so that $\mathrm{vol} = \gamma(x)dx_1 \wedge \cdots \wedge dx_n$ with $0 < \gamma(x) \in C^\infty$. Assume $\operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$ is relatively compact in some open set Ω. Integration by parts yields:

$$\begin{aligned}(P\alpha, \beta)_{L^2} &= \int_\Omega \sum_{|I|\leqslant k,i,j} P_{I,i,j}(x) D^I \alpha_j(x) \overline{\beta_i(x)} \gamma(x) dx \\ &= \int_\Omega \alpha_j(x) \sum_{|I|\leqslant k,i,j} (-1)^{|I|} \gamma^{-1}(x) \overline{D^I\big(\gamma(x)\overline{P_{I,i,j}(x)}\beta_i(x)\big)} \gamma(x) dx\end{aligned}$$

In which case

$$P^*\beta = \sum_{|I|\leqslant k,i,j} (-1)^{|I|} \gamma^{-1}(x) \overline{D^I\big(\gamma(x)\overline{P_{I,i,j}(x)}\beta_i(x)\big)}$$

is a unique local adjoint; extending by a partition of unity subordinate to some open cover, the result follows.

□

6. Results from Partial Differential Equations

In this section we will collect some standard results and proofs from the theory of PDE's in order to build up to one of the main results of the paper. It is assumed the reader is familiar with Sobolev spaces and the notation $W^{k,p}(\Omega)$. As is standard, define $H^k(\Omega) := W^{k,2}(\Omega)$; we are still assuming E is a smooth vector bundle over a smooth compact manifold M.

Lemma 6.1 (Sobolev's Lemma)**.** *Let $m \in \mathbb{N}$. For all $m > k + n/2$, where* $\operatorname{rank} E = n$*, we have a continuous inclusion*

$$\underline{H^k}(E) \hookrightarrow \underline{C^m}(E)$$

In particular,

$$\bigcap_{k \geqslant 0} \underline{H^k}(E) = \underline{C^\infty}(E)$$

Lemma 6.2 (Rellich-Kondrachov Compactness Theorem)**.** *Let $1 \leqslant r < \infty$ and let $j, m \in \mathbb{N}$ with $0 \leqslant j < m$. If $p \geqslant 1$ satisfies*

$$\frac{1}{p} > \frac{j}{n} + \frac{1}{r} - \frac{m}{n}$$

then the embedding

$$\underline{W^{m,r}}(E) \hookrightarrow \underline{W^{j,p}}(E)$$

is compact. In particular, whenever $m > j$,

$$\underline{H^m}(E) \hookrightarrow \underline{H^j}(E)$$

is compact.

Lemma 6.3 (Garding's Inequality)**.** *Let $P : \underline{C^\infty}(E) \to \underline{C^\infty}(F)$ be an elliptic differential operator of degree k with* $\operatorname{rank} E = \operatorname{rank} F = n$*. Suppose that $\widetilde{P}$ is an extension P. Then, for all $\alpha \in \underline{H^0}(E)$ such that $\widetilde{P}\alpha \in \underline{H^m}(F)$,*

$$\alpha \in \underline{H^{m+k}}(E)$$

and,

$$||\alpha||_{H^{m+k}} \leqslant C(m)\Big(||\widetilde{P}\alpha||_{H^m} + ||\alpha||_{H^0}\Big)$$

for some constant $C(m)$ depending only on m.

Finally, we will need the following theorem from Functional Analysis. In the below, B_X denotes the closed unit ball of X.

Lemma 6.4 (Riesz's Lemma)**.** *Suppose X is a Banach space and Y is a proper closed subspace of X. Given any $\epsilon > 0$, there exists some $x \in \partial B_X$ such that $d(x, Y) > 1 - \epsilon$. Furthermore, if Y is finite-dimensional, then $x \in \partial B_X$ can be chosen so that $d(x, Y) = 1$.*

As an immediate consequence of the above, we get

Corollary 6.5. *If X is an infinite dimensional Banach space, then B_X is not compact.*

7. The Finiteness Theorem

In this section we will present a fundamental theorem on finiteness properties of elliptic differential operators. This theorem is essential for the Hodge Decomposition.

Theorem 7.1. *Let E and F be Hermitian vector bundles on a compact manifold M with $\operatorname{rank} E = \operatorname{rank} F = n$.*

Suppose $P : \underline{C^\infty}(E) \to \underline{C^\infty}(F)$ is an elliptic differential operator of degree k. Then,

1. $\operatorname{Ker} P$ *is finite dimensional.*
2. $P\big(\underline{C^\infty}(E)\big)$ *is closed and of finite codimension. Moreover, if P^* is the formal adjoint of P, there exists a decomposition*

$$\underline{C^\infty}(E) = \operatorname{Ker} P \oplus P^*\big(\underline{C^\infty}(F)\big)$$

Proof. We first see that for any $\alpha \in \operatorname{Ker} P$, Garding's inequality gives that $||\alpha||_{H^{m+k}} \leqslant C(m)||\alpha||_{H^0}$.

By Sobolev's Lemma we deduce that $\operatorname{Ker} P$ must be closed in $\underline{H^0}(E)$, in which case we also see that $||\alpha||_{H^m} \leqslant C(0)||\alpha||_{H^0}$. This implies that the unit ball of $\operatorname{Ker} P$ is contained in the $C(0)$-ball of $\underline{H^m}(E)$. By the Rellich-Kondrachov compactness theorem, the inclusion of the unit ball of $\operatorname{Ker} P$ is a compact subset of $\underline{H^0}(E)$. By the corollary of Riesz's Lemma, we conclude that this implies $\dim \operatorname{Ker} P < \infty$, which proves part (1). □

For the proof of (2), we will divide this up into separate lemmas for sake of clarity.

Lemma 7.2. *Let $\alpha \in \operatorname{Ker} P$. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ and a set $\{\beta_1, \dots, \beta_N\} \subset \underline{H^{m+k}}(F)$ such that*

$$||\alpha||_{H^0} \leqslant \epsilon||\alpha||_{H^{m+k}} + \sum_{i=1}^{N} |\langle \alpha, \beta_i \rangle_{H^0}|$$

Proof. Let $\epsilon > 0$ and $\alpha \in \operatorname{Ker} P$. Choose elements $\beta_1, \dots, \beta_N \in \underline{H^0}(F)$ and consider the set

$$V_{\beta_j} := \{\alpha \in \underline{H^{m+k}}(F) \mid \epsilon||\alpha||_{H^{m+k}} + \sum_{i=1}^{N} |\langle \alpha, \beta_i \rangle_{H^0}| \leqslant 1\}$$

This is relatively compact in $\underline{H^0}(F)$ since the closure is contained in the unit ball of $\underline{H^0}(F)$, which is compact by the proof of (1). We also have that

$$\bigcap_{\beta_i} \overline{K_{\beta_j}} = \{0\}$$

in which case we deduce that we may choose our β_j to be contained in the unit ball of $\underline{H^0}(F)$. Thus the β_j satisfy the conditions of the statement. □

Lemma 7.3. *The extension $\widetilde{P}$ of P has closed image $\widetilde{P}\big(\underline{H^{m+k}}(E)\big)$ for all m.*

Proof. Choose $\alpha \in \operatorname{Ker} P$ as before. By the previous lemma, given $\epsilon > 0$ we have the inequality (for suitably chosen β_i)

$$||\alpha||_{H^0} \leqslant \epsilon ||\alpha||_{H^{m+k}} + \sum_{i=1}^{N} |\langle \alpha, \beta_i \rangle_{H^0}|$$

Now, substitute the above into Garding's inequality. We find:

$$(1 - C(m)\epsilon)||\alpha||_{H^{m+k}} \leqslant C(m)\Big(||\widetilde{P}\alpha||_{H^m} + \sum_{i=1}^{N} |\langle \alpha, \beta_i \rangle_{H^0}|\Big)$$

Now define $T := \operatorname{Span}\{\beta_1, \dots, \beta_N\}^{\perp}$. If we set $\epsilon = 1/2C(m)$, the above inequality yields

$$||\alpha||_{H^{m+k}} \leqslant 2C(m)||\widetilde{P}\alpha||_{H^m} \quad \text{for all } \alpha \in T$$

In which case we see that $\widetilde{P}(T)$ is closed. But then

$$\widetilde{P}\big(\underline{H^{m+k}}(E)\big) = \widetilde{P}(T) + \operatorname{Span}\{\widetilde{P}(\beta_1), \dots, \widetilde{P}(\beta_N)\}$$

is also closed, as desired. □

Finally, we may conclude the proof of the second statement in the finiteness theorem.

Proof of statement (2). Now, for $m = 0$, by definition $\underline{H^0}(E) = \underline{L^2}(E)$. Since our smooth sections are dense in every $\underline{H^m}(E)$ and the image of $\underline{H^k}(E)$ is closed, we see

$$\Big(\underline{H^k}(E)\Big)^{\perp} = \Big(P\big(\underline{C^\infty}(E)\big)^{\perp} = \operatorname{Ker} \widetilde{P}^*$$

whence we deduce

$$\underline{H^0}(E) = \widetilde{P}\big(\underline{H^k}(E)\big) \oplus \operatorname{Ker} \widetilde{P}^*$$

Note that the adjoint of any elliptic operator is also elliptic. By the proof of statement (1), this implies that $\operatorname{Ker} \widetilde{P}^*$ is also finite dimensional, so that $\operatorname{Ker} \widetilde{P}^* = \operatorname{Ker} P^*$ is contained in $\underline{C^\infty}(E)$. Applying

Garding's inequality, we then see

$$\underline{H^m}(E) = \widetilde{P}\big(\underline{H^{m+k}}(E)\big) \oplus \operatorname{Ker} P^*$$

And, employing the second statement of Sobolev's Lemma, we see

$$\underline{C^\infty}(E) = P\big(\underline{C^\infty}(E)\big) \oplus \operatorname{Ker} P^*$$

which completes the proof. □

8. Cohomology and Harmonic Forms

We begin with a simple proposition.

Proposition 8.1. *The Laplacian operator Δ_d is elliptic and self adjoint.*

Proof. Self adjointness is trivial; ellipticity follows from the fact that the Laplacian Δ_d has symbol

$$\sigma_{\Delta_d}(\alpha)(\omega) = -||\alpha||^2\omega$$

where $||\alpha||^2$ is equal to $(\alpha, \alpha)_x$ on fibers. □

Now, let (X, g) be a compact oriented Riemannian manifold; let $A^k(X)$ denote the smooth k-forms on X.

Theorem 8.2. *Let $\mathcal{H}^k$ denote the space of Δ_d-harmonic k-forms. The natural map*

$$\mathcal{H}^k \to H^k(X, \mathbb{R})$$
$$\alpha \mapsto [\alpha]$$

is an isomorphism; the same holds for the map from complex harmonic k-forms to $H^k(X, \mathbb{C})$.

Proof. The result of the previous section gives that

$$A^k(X) = \mathcal{H}^k \oplus \Delta\big(A^k(X)\big)$$

Choose $\beta \in A^k(X)$ a closed form. Then, $\beta = \alpha + \Delta\gamma$ for harmonic α; since β is closed we deduce that $d^*d\gamma$ must also be closed. However, $d^*d\gamma$ is clearly an element of $\operatorname{Im} d^*$, so $d^*d\gamma \in \operatorname{Ker} d \cap \operatorname{Im} d^* = \{0\}$. Whence

$$\beta = \alpha + dd^*\gamma \implies [\beta] = [\alpha]$$

giving surjectivity. Now assume β is both harmonic and exact. Since $\operatorname{Ker} \Delta_d = \operatorname{Ker} d \cap \operatorname{Ker} d^*$, we deduce that $\beta \in \operatorname{Ker} d^* \cap \operatorname{Im} d = \{0\}$, whence injectivity follows. □

In an identical manner, the following result for Dolbeault cohomology is proved:

Theorem 8.3. *Let E be a Hermitian holomorphic vector bundle over a complex manifold X equipped with a Hermitian metric. If $\mathcal{H}^{0,q}(E)$ is the space of harmonic forms of type $(0,q)$, the natural map*

$$\mathcal{H}^{0,q}(E) \to H^q(X, \rceil)$$

$$\alpha \mapsto [\alpha]$$

is an isomorphism.

Combining the above isomorphisms with the finite dimensionality guaranteed by the result of the previous section, we get

Corollary 8.4.
(1) *If X is a compact manifold, then the cohomology groups $H^q(X, \mathbb{R})$ are finite dimensional.*
(2) *If X is a compact manifold, the cohomology groups $H^q(X, E)$ are finite dimensional for every holomorphic vector bundle E over X.*

9. Kähler Identities and the Hodge Decomposition

Let X be a Kähler manifold with form ω. We can define the Lefschetz operator

$$\begin{aligned} L : & A^k(X) \to A^{k+2}(X) \\ & \alpha \mapsto \omega \wedge \alpha \end{aligned}$$

Let Λ denote the formal adjoint of L; we have

Proposition 9.1.

$$\Lambda = - *^{-1} L*$$

Proof. Let $\alpha \in A^k(X)$, $\beta \in A^{k+2}(X)$. Then,

$$\begin{aligned} (L\alpha, \beta)\mathrm{vol} &= \omega \wedge \alpha \wedge *\beta \\ &= -\alpha \wedge *\big(*^{-1} L * \beta\big) \\ &= (\alpha, - *^{-1} L * \beta)\mathrm{vol} \end{aligned}$$

□

Letting $[\cdot, \cdot]$ denote the commutator bracket, we also see

Proposition 9.2. *We have the identities*

$$[\Lambda, \overline{\partial}] = -i\partial^* \quad [\Lambda, \partial] = i\overline{\partial}^*$$

Theorem 9.3. *Let (X, ω) be a Kähler manifold, and let Δ_d, Δ_∂, and $\Delta_{\overline{\partial}}$ be the Laplacians associated to d, ∂, and $\overline{\partial}$, respectively. We have the relations*

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$$

Proof. We have

$$\begin{aligned} \Delta_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= (\partial + \overline{\partial})(\partial^* - i[\Lambda, \partial]) + (\partial^* - i[\Lambda, \partial])(\partial + \overline{\partial}) \\ &= \partial\partial^* + i\overline{\partial}\partial^* + i\overline{\partial}\partial\Lambda - i\overline{\partial}\Lambda\partial + \partial^*\partial + \partial^*\overline{\partial} - i\Lambda\partial\overline{\partial} + i\partial\Lambda\overline{\partial} \end{aligned}$$

Since $\partial^* = -[\Lambda, \overline{\partial}]$,

$$\partial^*\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial} = -\overline{\partial}\partial^*$$

Whence

$$\Delta_d = \Delta_\partial + i\partial[\Lambda, \overline{\partial}] + i[\Lambda, \overline{\partial}]\partial$$

$$\Delta_\partial + \partial\partial^* + \partial^*\partial$$

$$= 2\Delta_\partial$$

Similarly, we find

$$\Delta_d = \Delta_{\overline{\partial}} - i\overline{\partial}[\Lambda, \partial] - i\overline{\partial}[\Lambda, \partial]$$

$$= 2\Delta_{\overline{\partial}}$$

□

Combining the above, we get a series of quick yet powerful corollaries.

Corollary 9.4. *If X is Kähler, then*

$$\Delta_d\big(A^{p,q}(X)\big) \subset A^{p,q}(X)$$

That is, Δ_d is bihomogeneous.

Proof. Both Δ_∂ and $\Delta_{\overline{\partial}}$ are bihomogeneous, so the previous result yields the theorem. □

Corollary 9.5. *If $\alpha \in A^k(X)$ is harmonic, then each component $\alpha^{p,q}$ is also harmonic.*

Corollary 9.6. *We have a direct sum decomposition*

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

where $\mathcal{H}^{p,q}$ denotes the set of harmonic (p, q) forms.

Corollary 9.7. *We have a direct sum decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

Proof. We already have that $\mathcal{H}^k \cong H^k(X, \mathbb{C})$, and likewise for $\mathcal{H}^{p,q} \cong H^{p,q}$. Employing the previous corollary yields the result. $\square$

One may notice that the above corollary depends upon the choice of symplectic form ω for our Kähler manifold X, since this is how our Lefschetz operator is defined. In order for us to be able to refer to the above as *the* Hodge Decomposition, we need the following:

Proposition 9.8. *The decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

does not depend on the choice of Kähler metric.

Proof. Let $K^{p,q}$ denote the subspace of all cohomology classes that may be represented by a closed form of type (p, q). By definition, we have that $H^{p,q} \subset K^{p,q}$. We proceed to show the reverse inclusion.

Let α be a closed (p, q) form. In view of 7.1 and 8.2, we may write $\alpha = \beta + \Delta\gamma$ uniquely with β harmonic. By bihomogeneity of the Laplacian, both β and γ are also (p, q) forms. Since α is closed, we deduce that $d^*d\gamma = 0$, in which case $\alpha = \beta + dd^*\gamma$, so $[\alpha] = [\beta]$. But $[\beta] \in H^{p,q}(X)$, in which case we see $K^{p,q} = H^{p,q}(X)$.

Since $K^{p,q}$ is defined independently of the choice of metric, we see that $H^{p,q}(X)$ also does not depend on the metric. Whence the above decomposition is determined independent of the Kähler metric. $\square$

Combining the above with the previous corollary, we have proved the existence of the Hodge decomposition.

Corollary 9.9 (Hodge Decomposition)**.** *Let (X, ω) be a Kähler manifold. Then there exists a direct sum decomposition of homology groups*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

Moreover, this direct sum does not depend on the choice of metric ω.

Finally, as a final corollary we have the following:

Corollary 9.10 ($\partial\overline{\partial}$ Lemma)**.** *Let (X, ω) be a Kähler manifold, and suppose α is both ∂ and $\overline{\partial}$ closed. If α is d, ∂, or $\overline{\partial}$ exact, then there exists γ such that $\alpha = \partial\overline{\partial}\gamma$.*

Proof. Merely use the decomposition guaranteed by the finiteness theorem. □

References

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